

The vol smile problem

The comparative liquidity of vanilla and exotic options in the major forex markets makes them a useful experimental laboratory for testing volatility smile models. Here, Alexander Lipton examines a wide range of models in this context and identifies the variants that perform best

For many years, practitioners and academics have tried to analyse the volatility smile phenomenon and understand its implications for derivatives pricing and risk management. In a nutshell, the volatility smile means that vanilla options with different maturities and strikes have different implied Black-Scholes (1973) volatilities. Accordingly, accurate pricing and hedging of options cannot be achieved within the standard Black-Scholes framework. The objectives of the volatility smile studies are threefold: to find a model for pricing vanilla options across all strikes and maturities; to develop a reliable hedging strategy for these options; and to price and hedge exotic options consistently with vanilla options.

The purpose of this Masterclass article is to describe the state of the art in the pricing and hedging of options on foreign exchange rates from a financial engineer's perspective. These notes are accessible for quants, risk managers and traders alike. First, we describe a series of increasingly complex models that can be used to price and hedge vanilla options consistently with the market. We emphasise that, although all these models can be successfully calibrated to the market, they produce very different hedging strategies. Practically useful models have to yield distributions of the profit and loss that are sharply peaked at zero. Elsewhere we will show how to use the above models for pricing and hedging of path-dependent options, such as Asians, barriers, etc. It turns out that, for some exotic options, smile-consistent prices can be dramatically different from the corresponding Black-Scholes ones. Our main conclusion is that only models that take into account local, jump and stochastic features of the volatility dynamics and mix them in the right proportion are adequate for pricing and risk management of forex options.¹

Market overview

A recent survey by *Risk* of 12 leading forex dealers (*Risk* May 2001, pages 44–49; see also the Bank for International Settlements' 1998 survey for the overall market) showed that, in 2000, the total unadjusted notional volume of forex options was \$13,000 billion. The majority of these were plain vanilla calls and puts with short (less than one month) (38.5%), and medium maturities (one–18 months) (52.5%). These were followed by barrier options (4%), long-term vanilla options (maturities of more than 18 months) (1.5%), digitals (1%), Asians (0.7%) and more exotic products, such as volatility, variance and correlations swaps, and Parisian, various accrual-style, passport and some multi-currency options. The dominant 32% of the market are in dollar/yen options, with euro/dollar options coming a close second at 28.5%. A large percentage share of barrier and Asian options suggests that, in effect, a number of exotic options in the forex market are commoditised with significantly reduced margins. (A similar situation arises in the fixed-income market, where some exotic options, such as Bermudan swaptions, are highly commoditised.) Thus, building an adequate volatility smile model is necessary for successfully dealing in the competitive forex derivatives market.

Typical smile patterns

In view of its importance, we use the euro/dollar market as our main example. The exchange rate $S_E^{\$}$ represents the number of dollars one needs to pay to receive one euro. In general, when $S = A/U$, the numerator and

denominator currencies are called the accounting and underlying currencies, respectively. We start with the markets for vanilla options. These markets are characterised by their implied volatility matrices, $\sigma_i^{C,P}(T, \Delta)$, which encapsulate the pricing information for the most liquid calls, C, and puts, P. As a rule, this information is highly standardised: calls and puts are characterised by their deltas, Δ , rather than strikes, K, and only 10- Δ , 25- Δ out-of-the-money calls and puts and Δ -neutral straddles are considered. Moreover, to express market preferences directly, for a given maturity we introduce 10- Δ and 25- Δ risk reversals (RRs) and strangles (STRs):

$$RR(T, \Delta) = \sigma_1^C(T, \Delta) - \sigma_1^P(T, \Delta),$$

$$STR(T, \Delta) = \frac{1}{2}(\sigma_1^C(T, \Delta) + \sigma_1^P(T, \Delta)) - \sigma_1(T, \Delta\text{-neutral})$$

Broadly speaking, RRs and STRs are characteristics of the market skew and smile respectively. When RRs are positive (negative), the market sentiment favours the underlying (accounting) currency. The most liquid maturities are one-week, two-week, one-month, two-month, three-month, six-month, 12-month and 24-month. On occasion, other maturities and deltas are considered. A typical volatility matrix is given in table A.

At this stage, we have to confront the first problem: the market data is too sparse. This problem is resolved via interpolation. A number of interpolation schemes have been proposed. Usually, cubic splining in the Δ -direction and linear splining in the T-direction are adequate. One aspect that needs to be addressed is a proper choice of the asymptotic behaviour of volatility for very small deltas. At present, one can confidently assume that dealers have reliable implied volatility surfaces $\sigma_1(T, \Delta)$ (or $\sigma_1(T, K)$) extending all the way to $1\% < \Delta < 50\%$, $1\text{-week} < T < 2\text{-year}$. The volatility surface corresponding to table A is shown in figure 1. In principle, we can think of σ_1 as a function of four arguments, $\sigma_1(t, S, T, K)$, even though at any given time we can see only its projection, $\sigma_1(t, S_t, T, K)$. An adequate choice of this function is a matter of much debate and has crucial risk management implications.

Daily variations of the spot rate and the corresponding volatility parameters are very significant. Figure 2 shows the behaviour of the Δ -neutral volatilities, RRs and STRs in the euro/dollar market as functions of calendar time for one year. This variability attracted a considerable attention of econometricians who developed various schemes to capture its idiosyncrasies. In the euro/dollar market, there is a strong positive correlation between the spot and RRs.

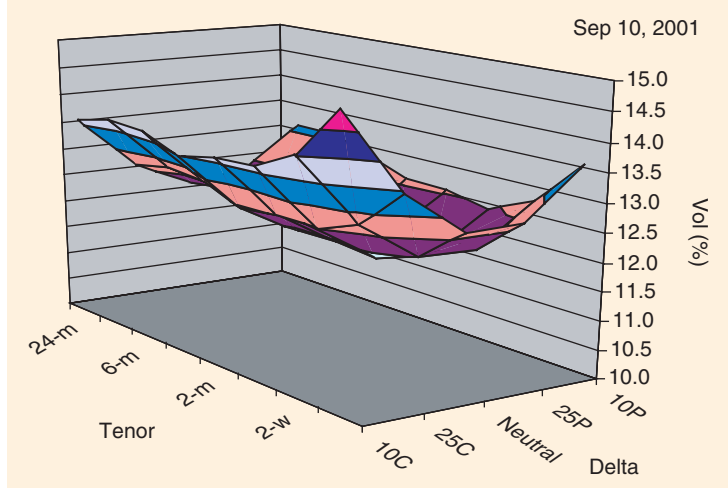
A. Euro/dollar market: September 10, 2001

T\Δ	10C	25C	Neutral	25P	10P	Euro	Dollar
1-week	14.89	13.46	12.70	12.72	13.55	4.35	3.56
2-week	14.02	12.76	12.07	12.05	12.75	4.35	3.54
1-month	13.66	12.49	11.83	11.80	12.41	4.34	3.49
2-month	13.63	12.52	11.91	11.92	12.56	4.28	3.40
3-month	13.48	12.43	11.88	11.94	12.59	4.24	3.35
6-month	13.73	12.71	12.11	12.22	12.85	4.10	3.32
12-month	13.78	12.80	12.26	12.41	13.08	3.95	3.43
24-month	13.61	12.66	12.20	12.38	13.10	4.01	3.99

Spot rate $S_E^{\$}$ is 0.8998

¹ A detailed account of the volatility smile problem in the forex context, including an extensive bibliography, is given in a recent book by Lipton (2001)

1. Typical euro/dollar volatility surface



One useful source of information encapsulated in the volatility surface is the implied distribution of the forex rates for a given maturity. This distribution is given by the formula due to Breeden & Litzenberger (1978), which can be derived by twice differentiating the usual risk-neutral valuation formula for a call option with respect to its strike:

$$p(T, K) = e^{r^0 T} \partial_{KK}^2 \{C(T, K)\} \quad (1)$$

We show probability density functions (PDFs) for one-month rates in figure 3. For comparison, we also show Gaussian PDFs with the corresponding at-the-money volatility. Even a cursory inspection of figure 3 reveals that the implied distributions have fatter tails and sharper peaks compared with the reference normal distributions. In other words, they are leptokurtic. It is not difficult to manufacture such distributions for a given maturity (for instance, a mixture of independent normal distributions can be used). However, to do so in a systematic way across all maturities is much more difficult.

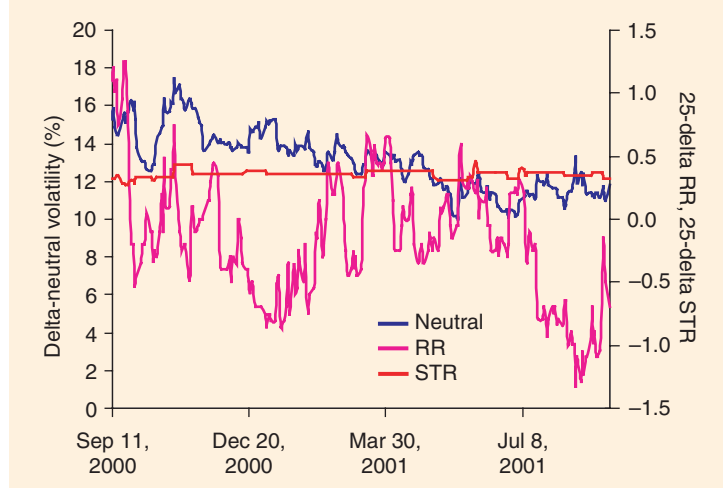
Generalisations of Black-Scholes: possibilities and pitfalls

The presence of skews, smiles and, to a lesser degree, term structures violates the most basic assumptions of the Black-Scholes model and makes it necessary to revisit the concept of pricing and hedging of vanilla options. Thus, to accommodate market reality, it is necessary to extend the Black-Scholes model in a meaningful fashion. In particular, one needs to generate leptokurtic distributions via a stochastic process for the spot and possibly some additional hidden variables. The main difficulty is that there are many processes that can be used for this purpose, and their relative merits and drawbacks partly depend on a specific problem at hand.

A number of models have been proposed in the literature: the local volatility models of Dupire (1994), Derman & Kani (1994) and Rubinstein (1994); a jump-diffusion model of Merton (1976); stochastic volatility models of Hull & White (1988), Heston (1993) and others; mixed stochastic jump-diffusion models of Bates (1996) and others; universal volatility models of Dupire (1996), JP Morgan (1999), Lipton & McGhee (2001), Britten-Jones & Neuberger (2000), Blacher (2001) and others; regime switching models, etc. Too often, these models are chosen *ad hoc*, for instance, on the grounds of their tractability and solvability. However, the right criterion, as advocated by a number of practitioners and academics, is to choose a model that produces hedging strategies for both vanilla and exotic options resulting in profit and loss distributions that are sharply peaked at zero. (Recall that in the classical Black-Scholes model such a distribution is described by a delta function.)

□ **Black-Scholes models.** The stochastic differential equation (SDE) that describes the standard geometrical Brownian motion (GBM) for S_t and is used in the Black-Scholes (1973) theory is:

2. Behaviour of the one-month smile parameters in the euro/dollar market



$$\frac{dS_t}{S_t} = r^{01} dt + \sigma dW_t \quad (2)$$

where $r^{01} = r^0 - r^1$, and r^0, r^1 are interest rates in the accounting and underlying currencies, respectively, and σ is the forex volatility. The pricing problem for a call has the form:

$$C_t^{(BS)} + \frac{1}{2} \sigma^2 S_t^2 C_{SS}^{(BS)} + r^{01} S_t C_S^{(BS)} - r^0 C^{(BS)} = 0, \\ C^{(BS)}(T, S) = (S - K)_+$$

The price $C^{(BS)}$ is given by the formula:

$$C^{(BS)} = e^{-r^1 T} \text{SN}(d_+) - e^{-r^0 T} \text{KN}(d_-)$$

where $d_{\pm} = X/\sigma\sqrt{T} \pm \sigma\sqrt{T}/2$, $X = (\ln(S/K) + r^{01}T)$. For the purposes of comparison with more general pricing formulas given below, it is more convenient to write $C^{(BS)}$ as a Fourier integral:

$$C^{(BS)}(0, S, T, K) = e^{-r^1 T} S - \frac{e^{-r^0 T} K}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(-ik+1/2)X - (k^2+1/4)\sigma^2 T/2}}{(k^2+1/4)} dk \quad (3)$$

which can be derived by representing the payout of a call in the form:

$$(S - K)_+ = S - \min\{K, S\}$$

and dealing with the bounded component of the payout.

As we will see below, all exactly solvable models generate prices that are simple generalisations of (3). We note that an interesting alternative approach to the derivation of formulas similar to (3) has been developed by Bakshi & Madan (2000), who used the apparatus of characteristic functions and gave a spanning interpretation to the Fourier transform of the state prices.

□ **Local volatility models.** The most straightforward approach is to generalise the governing SDE for S_t :

$$\frac{dS_t}{S_t} = r^{01} dt + \sigma_L(t, S_t) dW_t$$

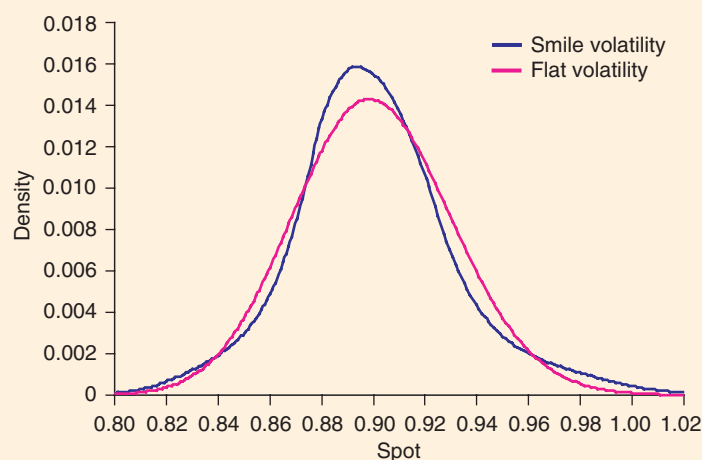
where the local volatility $\sigma_L(t, S)$ is a deterministic function of its arguments. The pricing equation for a European-style call, say, is:

$$C_t^{(LV)} + \frac{1}{2} \sigma_L^2(t, S) S_t^2 C_{SS}^{(LV)} + r^{01} S_t C_S^{(LV)} - r^0 C^{(LV)} = 0$$

Since there is just one source of uncertainty, this model is complete and all options can be perfectly delta-hedged.

In general, for a given local volatility one has to solve the pricing problem numerically. However, in some cases, such as the term structure (TS) model, $\sigma_L^{(TS)} = \sigma(t)$, the constant elasticity of variance (CEV) model, $\sigma_L^{(CEV)} = \sigma_0(S/S_0)^{-\rho}$ and the hyperbolic (H) volatility model:

3. PDF for one-month euro/dollar forex rate



$$RR(T, \Delta) = \sigma_1^C(T, \Delta) - \sigma_1^P(T, \Delta), \quad (4)$$

it can be solved analytically. These analytical solutions are useful for illustrative purposes.

The most important qualitative feature of a local volatility model is that the smile changes with the spot, or, in other words, that σ_1 depends on both S and K , rather than on the moneyness S/K alone. To put it differently, even if initially such a model generates a smile, as spot moves it will necessarily generate a skew. This fact has important hedging implications. For local volatility models, the magnitude of the corresponding Δ is different from the one observed in the market.

To illustrate the above observations, we consider the hyperbolic volatility model. The hyperbolic volatility model with positive roots was analysed by Ingersoll (1996) and Rady (1997) and the general case was solved by Zuhlsdorff (1999) and Lipton (2000). To be concrete, we assume that $\varpi = \beta^2 - 4\alpha\gamma > 0$, $\beta > 0$, so that the equation $\alpha x^2 + \beta x + \gamma = 0$ has real and negative roots, and denote them by p, q , $p < q < 0$. The corresponding $C^{(LV)}$ can be found via the Fourier series method:

$$\begin{aligned} C^{(LV)}(0, S, T, K) &= e^{-r^0 T} S - e^{-r^0 T} K \frac{2(\alpha \vartheta^2 + \beta \vartheta + \gamma)^{1/2} (\alpha + \beta/\chi + \gamma/\chi^2)^{1/2}}{\varpi^{1/2} \ln(p/q)} \\ &\times \sum_{n=1}^{\infty} \frac{E_n}{(k_n^2 + 1/4)} \sin\left(k_n \ln\left(\frac{\xi - p}{\xi - q}\right)\right) \sin\left(k_n \ln\left(\frac{\chi - p}{\chi - q}\right)\right) \\ \vartheta &= S/S_0, \quad \chi = K/e^{r^0 T} S_0, \quad k_n = \pi n / \ln(p/q), \quad E_n = e^{-(k_n^2 + 1/4)\varpi \sigma_0^2 T/2} \end{aligned} \quad (5)$$

Expression (5) can be viewed as a generalisation of (3).

□ **Jump diffusion models.** As a rule, short-dated options exhibit a strong smile that is difficult to account for by using continuous diffusion processes. Besides, empirical research by Bates (1996) and others suggests that a possibility of sudden large jumps in Group of Seven exchange rates, such as the one that followed the Plaza Accord, has an important impact on market sentiment. In addition, sudden devaluations of emerging markets currencies are not uncommon. Because of all these factors, jumps have to be incorporated into the picture. The governing SDE for S_t becomes:

$$\frac{dS_t}{S_t} = (r^0 - \lambda m) dt + \sigma dW_t + (e^j - 1) dN_t$$

where N_t is a Poisson process with frequency λ , which is independent of the Wiener process W_t , $j > 0$ is a random logarithmic jump size with PDF $\phi(j)$ and m is the expected value of $e^j - 1$. We emphasise that, in contrast to the

local volatility model, this model is incomplete. Although the risk-neutral drift has to have the form $(r^0 - \lambda m)$, the values of λ and m are not uniquely given by the price dynamics. In principle, these values can be inferred from option prices. The corresponding pricing problem for a call becomes:

$$\begin{aligned} C_t^{(JD)} + \frac{1}{2} \sigma^2 S^2 C_{SS}^{(JD)} + (r^0 - \lambda m) S C_S^{(JD)} \\ + \lambda \int C^{(JD)}(e^j S) \phi(j) dj - (r^0 + \lambda) C^{(JD)} = 0 \end{aligned}$$

Qualitatively, jump diffusion models produce distributions of returns that are mixtures of normal distributions and do have attractive leptokurtic features, at least for short maturities. By construction, the jump diffusion implied volatility depends only on the moneyness, $\xi = S/K$, and, hence, is relative in nature. Thus, if a model generates a smile initially, it will continue to generate a smile when spot moves.

Merton (1976) showed that for the lognormal distribution of jump sizes it is possible to represent the price of a call as a weighted average of the standard Black-Scholes prices. In general, the pricing problem has to be solved via the Fourier transform method. We generalise (3) and represent the jump diffusion price as:

$$\begin{aligned} C^{(JD)}(0, S, T, K) &= e^{-r^0 T} S - \frac{e^{-r^0 T} K}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(-ik+1/2)X + \alpha^{(JD)}(T, k) - (k^2 + 1/4)\sigma^2 T/2}}{(k^2 + 1/4)} dk, \\ \alpha^{(JD)} &= \lambda T \left[\int_{-\infty}^{\infty} e^{(-ik+1/2)j} \phi(j) dj - 1 - m(ik + 1/2) \right] \end{aligned}$$

For short maturities, this surface has a profound skew that rapidly disappears when maturity increases. When applicable, Merton's infinite sum representation is easier to deal with than the general Fourier representation.

□ **Stochastic volatility models.** Figure 2 suggests that the at-the-money volatility behaves randomly. Accordingly, we need to model its evolution as a stochastic process, on a par with the evolution of the forex rate itself. For illustrative purposes, we choose the popular Heston (1993) model with mean-reverting volatility and assume that:

$$\frac{dS_t}{S_t} = r^0 dt + \sqrt{v_t} dW_t^{(S)}, \quad dv_t = \kappa(\theta - v_t) dt + \varepsilon \sqrt{v_t} dW_t^{(v)}$$

where $W_t^{(S)}, W_t^{(v)}$ are two correlated Wiener processes with correlation ρ . All stochastic volatility models are incomplete. To avoid a lengthy discussion of the market price of risk, we simply consider their risk-neutralised versions.

The corresponding pricing equation for a call is:

$$\begin{aligned} C_t^{(SV)} + \frac{1}{2} v \left(S^2 C_{SS}^{(SV)} + 2\rho \varepsilon S C_{Sv}^{(SV)} + \varepsilon^2 C_{vv}^{(SV)} \right) + r^0 S C_S^{(SV)} \\ + \kappa(\theta - v) C_t^{(SV)} - r^0 C^{(SV)} = 0 \end{aligned}$$

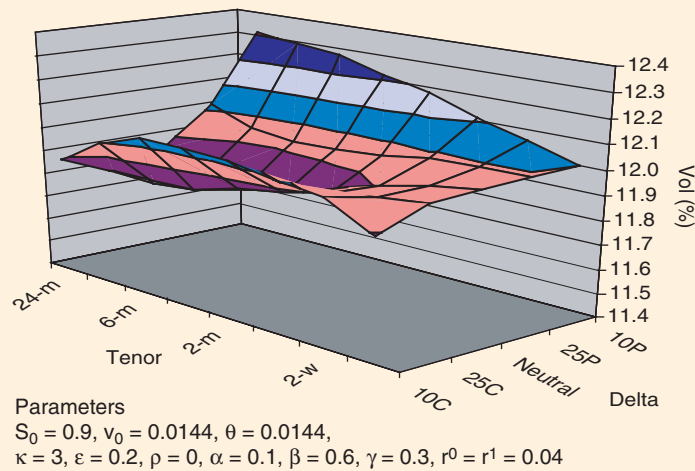
It is clear that the implied volatilities generated in the stochastic volatility framework are relative in nature. Accordingly, the smile parameters (RRs and STRs) do not change with the spot, but do change with the instantaneous volatility. Such models are capable of producing a rich variety of smile and term structure patterns. Compared with jump diffusion models, these patterns are much less profound for small maturities but tend to persist for longer ones. Stochastic volatility models can easily fit the forex markets, except for very short maturities. Unfortunately, experience suggests that the corresponding hedging strategies are not perfect and tend to produce profit and loss distributions that are not sufficiently sharply peaked.

A simple and intuitive way of understanding the meaning of stochastic volatility models is to view them as vehicles for averaging of Black-Scholes prices with respect to volatility and possibly spot values. For zero correlation, $\rho = 0$, a simple scenario analysis yields the following exact representation:

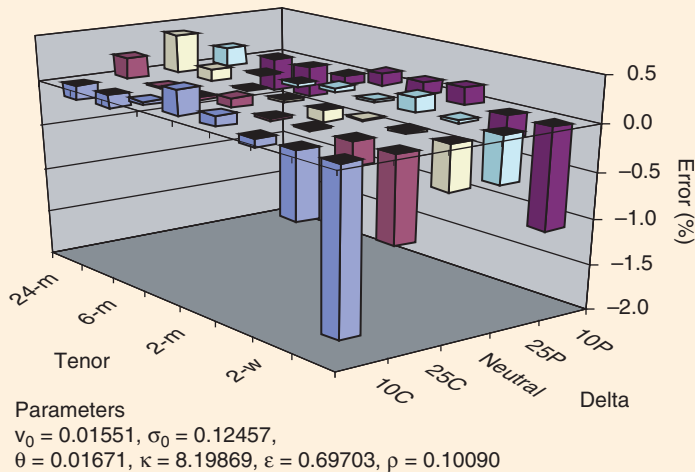
$$C^{(SV)} = \int C^{(BS)}(S, \hat{\sigma}) f(\hat{\sigma}) d\hat{\sigma}, \quad \hat{\sigma} = \sqrt{\sigma^2 dt/T}$$

where $\hat{\sigma}$ is the average volatility (Hull & White, 1988). For non-zero correlation, we can write S_t as a product of two random processes, one of which is uncorrelated with v and the other is perfectly correlated with it, and represent $C^{(SV)}$ as:

4. 'Universal' implied volatility



5. Quality of calibration for the stochastic volatility model



$$C^{(SV)} = \int C^{(BS)} \left(e^{-\rho^2 \hat{\sigma}^2 T / 2 + \rho J S} \sqrt{1 - \rho^2 \hat{\sigma}^2} \right) g(\hat{\sigma}, J) d\hat{\sigma} dJ, \quad J = \int \sigma dW$$

(Willard, 1997). Even though it is difficult to find the PDFs of f and g , the above formulas are still very useful since they reduce the dimensionality of the problem and allow one to price vanilla options efficiently via a one-dimensional Monte Carlo method. For $\rho = 0$, when only the volatility is averaged, the smile is necessarily symmetric. The approximate correction to the Black-Scholes price is proportional to ε^2 and is positive when $d_+ d_- > 1$. For $\rho \neq 0$, when both the volatility and spot are averaged, the skew that arises in a natural way is a dominant factor. The corresponding price correction is proportional to $\rho \varepsilon$ and is positive when $\rho d_+ > 0$.

To obtain a more detailed picture, we solve the pricing problem directly via the Fourier transform method. The corresponding price is:

$$C^{(SV)}(0, S, v, T, K) = e^{-r^1 T} S - \frac{e^{-r^0 T} K}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(-ik + 1/2)X + \alpha^{(SV)}(T, k) - (k^2 + 1/4)\beta^{(SV)}(T, k)v}}{(k^2 + 1/4)} dk \quad (6)$$

$$\alpha^{(SV)} = -\frac{\kappa \theta}{\varepsilon^2} \left[\psi_+ \tau + 2 \ln \left(\frac{\psi_- + \psi_+ e^{-\zeta \tau}}{2\zeta} \right) \right], \quad \beta^{(SV)} = -\frac{1 - e^{-\zeta \tau}}{\psi_- + \psi_+ e^{-\zeta \tau}} \quad (7)$$

$$\psi_{\pm} = \mp (ik\rho\varepsilon + \hat{\kappa}) + \zeta, \quad \zeta = \sqrt{k^2\varepsilon^2(1 - \rho^2) + 2ik\rho\varepsilon\hat{\kappa} + \hat{\kappa}^2 + \varepsilon^2/4}$$

where $\hat{\kappa} = \kappa - \rho\varepsilon/2$. This formula is much better suited for the purposes of numerical integration than its more popular standard counterpart because the integrand is non-singular. A convenient way to evaluate the corresponding integral is via the fast Fourier transform.

□ **Universal volatility models.** To account for the local, jump and stochastic features of the spot and its volatility, we need to combine all the above models and consider the following dynamics for the pair S, v :

$$\begin{aligned} \frac{dS_t}{S_t} &= (r^{01} - \lambda \vartheta) dt + \sqrt{v_t} \sigma_L(t, S_t) dW_t^{(S)} + (e^j - 1) dN_t, \\ dv_t &= \kappa(\theta - v_t) dt + \varepsilon \sqrt{v_t} dW_t^{(v)} \end{aligned} \quad (8)$$

It is also easy to add jumps in volatility but for our purposes it is not necessary. The corresponding pricing equation is:

$$\begin{aligned} C_t^{(UV)} + \frac{1}{2} v \left(\sigma_L^2(t, S) S^2 C_{SS}^{(UV)} + 2\rho\varepsilon\sigma_L^2(t, S) S C_{vS}^{(UV)} + \varepsilon^2 C_{vv}^{(UV)} \right) \\ + (r^{01} - \lambda m) S C_S^{(UV)} + \kappa(\theta - v) C_v^{(UV)} \\ + \lambda \int_{-\infty}^{\infty} C^{(UV)}(e^j S) \phi(j) dj - (r^0 + \lambda) C^{(UV)} = 0 \end{aligned} \quad (9)$$

This model is very rich and generates implied volatilities with both absolute and relative features. Moreover, it allows one to fine-tune the magnitudes of ε, λ , etc, and to achieve a proper mix of local, jump and stochastic features of the problem.

Unfortunately, it is very difficult to find explicitly solvable universal models. While combining jump diffusions and stochastic volatilities is straightforward since both are relative in nature, mixing stochastic and local volatilities requires a considerable effort. We assume that $\sigma_L = \sigma_L^{(H)}$, where $\sigma_L^{(H)}$ is given by (4), while $\rho = 0$. The price $C^{(UV)}$, which can be found via a combination of the Fourier series method and an affine ansatz, has the form (5), with:

$$E_n = e^{\alpha^{(SV)}(\tau, k_n) - (k^2 + 1/4)\beta^{(SV)}(\tau, k_n)v}$$

where $\alpha^{(SV)}, \beta^{(SV)}$ are given by formulas (7) with $\rho = 0$ and $\varepsilon = \varpi^{1/2}\varepsilon$. Figure 4 shows the implied volatility for a representative choice of parameters.

Calibration

In the previous section, we showed how to calculate prices and implied volatilities of vanilla options for given parameters characterising local, jump and stochastic components. However, there is no guarantee that our choice of these parameters is compatible with the market. To ensure that this is the case, we need to solve the calibration problem. From a mathematical standpoint, it is an ill-posed and unstable inverse problem. This fact is illustrated by the classical formula that connects the implied and local volatilities in the presence of the term structure:

$$\sigma_L(T) = \sqrt{d(\sigma_1^2(T)T)/dT} \quad (10)$$

It is clear that successful application of this formula depends on our knowledge of σ_1 for all maturities and on its good behaviour.

A naive approach to calibration is based on calculating prices of options for the most liquid strikes and maturities on an individual basis and modifying the parameters of the model until there is a match with the market. Unless analytical pricing formulas are available, is a very tedious and time-consuming task. However, when such formulas are known, this approach can be very efficient. Figure 5 shows the quality of a typical calibration to the market based on formula (6) for the price of a call option in the stochastic volatility framework. It is clear that the pure stochastic volatility model cannot handle very short maturities properly, otherwise the quality of calibration is respectable. By introducing jumps and term structure of parameters, it is possible to improve the quality of calibration by an order of magnitude.

Fortunately, by using the Fokker-Planck equation for the PDF $P(0, S, v, T, K, w)$, where (S, v) and (K, w) are the spot and variance values at times 0 and T respectively, combined with the Bredeen & Litzenberger formula (1), we can greatly accelerate the calibration procedure by pricing all the relevant calls at once. The Fokker-Planck problem corresponding to the general process (8) has the form:

$$P_T - \frac{1}{2} (w \sigma_L^2(T, K) K^2 P)_{KK} - (\rho \epsilon w \sigma_L(T, K) K P)_{wK} - \frac{1}{2} (\epsilon^2 w P)_{ww} + \left((r^{01} - \lambda m) K P \right)_K + (\kappa(\theta - w) P)_w - \lambda \int P(e^{-j} K) e^{-j} \phi(j) dj + \lambda P = 0, \quad (11)$$

$$P(0, K, w) = \delta(S - K) \delta(v - w)$$

We introduce the unconditional PDF:

$$Q(0, S, v, T, K) = \int_{-\infty}^{\infty} P(0, S, v, T, K, w) dw$$

and integrate the Fokker-Planck problem to obtain:

$$Q_T - \frac{1}{2} (v(T, K) \sigma_L^2(T, K) K^2 Q)_{KK} + \left((r^{01} - \lambda m) K Q \right)_K - \lambda \int Q(e^{-j} K) e^{-j} \phi(j) dj + \lambda Q = 0,$$

$$Q(0, K) = \delta(S - K)$$

where v is the conditional stochastic variance, which is defined as follows:

$$v(T, K) = \frac{\int_{-\infty}^{\infty} v P(T, K, v) dv}{Q(T, K)}$$

The Bredeen-Litzenberger formula relates Q and C as follows:

$$Q(T, K) = e^{r^{0T}} C_{KK}(T, K), \quad Q(T, e^{-j} K) = e^{2j} e^{r^{0T}} C_{KK}(T, e^{-j} K)$$

We use this relation to get the following forward equation for C :

$$C_T - \frac{1}{2} v(T, K) \sigma_L^2(T, K) K^2 C_{KK} + (r^{01} - \lambda m) K C_K - \lambda \int C(e^{-j} K) e^{-j} \phi(j) dj + (r^1 + \lambda(m+1)) C = 0, \quad (12)$$

$$C(0, K) = (S - K)_+$$

This equation allows us to price all the relevant calls in one sweep. In different special cases, it was derived by Dupire (1996), Anderson & Andreasen (2000) and others. However, an explicit combination of equations (11) and (12) seems to be new.

We can use equation (12) to get:

$$\sigma_L^2(T, K) = \frac{2 \left[C_T + (r^{01} - \lambda m) K C_K - \lambda \int C(e^{-j} K) e^{-j} \phi(j) dj + (r^1 + \lambda(m+1)) C \right]}{v(T, K) K^2 C_{KK}(T, K)} \quad (13)$$

In principle, the two-dimensional information encapsulated in equation (13) is sufficient to calibrate $\sigma_L^2(T, K)$ to the market. However, in practice, the calibration is very involved.

When the volatility is deterministic, so that $v(T, K) = \sigma_0^2$, equation (13) defines the local volatility similar to (10). However, since the calculation requires interpolating the original implied volatility matrix and involves numerical differentiation, it is prone to instabilities and has to be avoided. A better alternative is to choose a particular functional form for σ_L , such as cubic-linear spline, and to use equation (12) to price liquid vanilla options.

It is relatively easy to incorporate jumps into the picture, as was done by Anderson & Andreasen (2000). However, it is difficult to handle the stochastic component efficiently. The simplest approach is to replace equations (12) and (13) by their explicit finite difference approximations and to perform the forward induction in the spirit of Jamshidian (1991). A conceptually similar method based on the forward Markov chain approximation was proposed by Britten-Jones & Neuberger (2000). This tree-like approach tends to be numerically unstable and in practice should be replaced by a hybrid approach with both explicit and implicit features (Lipton & McGhee, 2001). However, the latter approach cannot be described here due to the lack of space. ■

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